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Steady-state availability of a multi-element symmetric pair

K e y w o r d s

Semi-Markov process, multi-element symmetric pair, steady-state availability calculation.

S ł o w a k l u c z o w e

Proces semi-Markowa, wieloelementowa para symetryczna, graniczny współczynnik gotowości.

S u m m a r y

This article is a continuation of the consideration in the study "Semi-Markov process for a pair of elements," published in 2011. However, this paper discusses the issues concerning a multi-element symmetric pair and not just a pair of elements alone. The problems with the operation of this type of system are identified and discussed. An analysis of this multi-element system was done with a proposed modelling method of its operation. The presented methods take into account different cases, depending on the result of an empirical study, exactly the type of probability distributions of times of elements states which the system are composed. The modelling method provides solutions that allow the basic reliability parameters of the multi element system to be obtained.

Introduction

There are a number of technical systems consisting of one basic element and a second one held in reserve. There are also a number of such pairs in

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mining engineering, e.g. a one-belt conveyor carrying a valuable mineral to its destination and a second conveyor serving as a spare. Such a solution makes sense if the stream of the mineral is high. Generally, there are five problems associated with the operation of a system of this kind.

At the very beginning is the problem:

(a) Whether it is an economically rational decision to add a spare unit to operating element?

Further problems are typically operational ones, namely:

(b) Which method of system utilisation should be selected?
(c) What is the intensity of the failures of spare element?
(d) Which mathematical model should be applied to adequately describe the operation of the system and what is the best way to assess the basic system parameters?

The last concern that can be associated with some generalisations is as follows:

(e) Each unit of the system consists of a certain number of elements connected in series.

To obtain an answer to the first question – applying the necessary economic considerations – it is necessary to get answers to questions (b) to (d), keeping in mind that the last point should be taken into account if the system is of such a structure.

Consider problem (b). Analysing the reliability of pairs of elements operating in several different technical fields, around fifty years ago, engineers discovered that the problem of the manner of system utilisation is important. As a rule, three different methods of system operation were taken into account, that is a "symmetric pair," a "pair in order" and a "pair half-loaded" (e.g. Czaplicki 2010, Chapter 7).

The operation order for a symmetric pair is as follows. One element executes its duties, and the second one is in a cold-type reserve. When a failure occurs in the working element, the second element commences its duties without delay. The first element is then in a repair state. When the repair is finished – a renewal occurs – the first element becomes the reserve. This situation exists until the moment when a failure occurs in the second element. The situation is then reversed. A failure of the system occurs when a failure occurs in the working element during the repair of the other.

Mining practice has shown that this method of operation of the system is most convenient or its equivalent – a pair in which elements are switched into operation deterministically from time to time, not waiting for a failure to occur. This is important for mechanical elements; but for electronics items, it may be invalid. In mechanical devices, some unwanted processes may occur during their long standstill, e.g., fluids sediments gathering at bottom, extensive belt sag, etc.

Presume here that the system of interest is operating under the "symmetric pair" regime, i.e. both elements are used uniformly in a long run in a stochastic sense. Thus, for a discussion of reliability, it does not matter whether the system can be treated as a symmetric pair or a pair deterministically switched on-and-off on a time schedule.

This problem (c) was discussed at the very beginning of the problem formation (for example: Gnyedenko 1964, 1969, Gnyedenko et al. 1965, Kopociński 1973). Each unit can be in its two own states, namely, a work and repair, and one state being a result of the system construction – standstill/reserve. Now, the problem is whether an element can or cannot fail when it is in a standstill state. The possibility of a failure in a spare part was one of the preliminary assumptions, making the reserve a "warm type." Some results were presented in the cited papers in connection with different types of reserves; however, they were mainly in the shape of Laplace transforms. Although, it looks like the most important result, especially from a practical point of view, it is based on the assumption of a "cold type" reserve. Usually, the intensity of the failures of an element in reserve is none or very small and, for this reason, can be neglected.

Method of system modelling

A first step in an analysis of system operation is the choice of the method of modelling its operation. If an empirical investigation shows that the probability distributions of the times of element states can be described by exponential distributions, the whole system operates according to the Markov process. This is the simplest case and the corresponding process of changes of states for the system is well known and was recently recalled by Czaplicki (2010 Chapter 7). However, in many practical cases, this assumption does not hold. The times of states are usually independent of each other but their probability distributions are not exponential, or only one is exponential. If so, the process of the changes of the states of the system can be described by a semi-Markov process. Let us discuss such an option in detail.

The method of analysis of such a system developed in the Markov process can be restated as follows: We start from an analysis of a series system, and when the appropriate characteristic functions are obtained, we consider a pair of elements.

Analysis of a series system

Consider an exploitation repertoire[1] $\{\mathfrak{S}_i;\ i=1,2,\ \ldots,n+1\}$ for states of the system. It consists of $n+1$ states; n states of repair because the system has n

[1] An exploitation repertoire is a defined set of the possible states of a given object, i.e. states in which object can be.

elements and *1* state of work. Denote the set of the repair states of the system by (**0**) and the set of the work states of the system by (**1**).

If so, an exploitation graph for the process of changes of states can be illustrated as is shown in Fig. 1a. In this figure, information on the corresponding probabilities of a transition between states is given. Figure 1b, in turn, shows the principle of passages between the states with information on the probability distributions $\hat{Q}_{ij}(t)$ concerning the transition from state *i* to state *j*.

Notice:

(i) $Q_{ij}(t)$; $i,j=1,2,\ldots,n+1$ denotes the probability distribution of time that process stay in state *i* and will jump to state *j*.

(ii) In all cases, one subscript is *1* because system analyzed is a series one.

(iii) The probability distributions $\hat{Q}_{ij}(t)$ are determined by the equations

$$\hat{Q}_{ij}(t) = p_{ij} Q_{ij}(t); \qquad \bigwedge_i p_{i1} \equiv 1 \tag{1}$$

(iv) Obviously, $\sum_{j=2}^{n+1} p_{1j} = 1$.

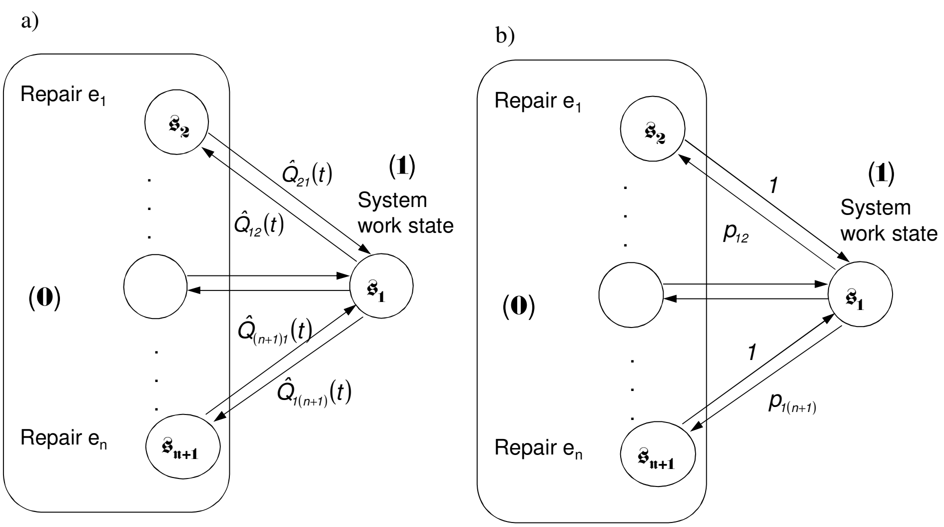

Fig. 1. An exploitation graph of the process of the changes of states for a series system with: a) probabilities of transition between states, b) probability distributions of passages between states
Rys. 1. Graf eksploatacyjny dla procesu zmiany stanów systemu szeregowego: a) prawdopodobieństwa przejść pomiędzy stanami, b) rozkłady prawdopodobieństw przejść pomiędzy stanami

Now we can construct the embedded Markov chain for the semi-Markov process of the system. We have the following:

$$\mathfrak{P} = \begin{pmatrix} P_{00} & P_{10} \\ P_{01} & P_{11} \end{pmatrix} = \begin{pmatrix} \boxed{0} & \boxed{p_{12} \ \cdots \ p_{1(n+1)}} \\ \boxed{\begin{matrix} 1 \\ \cdots \\ 1 \end{matrix}} & \boxed{\begin{matrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{matrix}} \end{pmatrix} \tag{2}$$

Based on the total probability principle, the following equations hold:

$$\begin{aligned} p_{12} &= \int_0^\infty (1-Q_{13}(t))\dots(1-Q_{1(n+1)}(t))dQ_{12}(t) \\ &\quad \cdots \\ p_{1(n+1)} &= \int_0^\infty (1-Q_{12}(t))\dots(1-Q_{1n}(t))dQ_{1(n+1)}(t) \end{aligned} \tag{3}$$

If so, the semi-Markov kernel is given by equation:

$$\mathfrak{Q}(t) = \begin{pmatrix} 0 & \hat{Q}_{12}(t) & \dots & \hat{Q}_{1(n+1)}(t) \\ \hat{Q}_{21}(t) & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \hat{Q}_{(n+1)1}(t) & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

It is a characteristic feature of series systems that non-zero elements besides the first element are only in the first row and in the first column.

The ergodic probability distribution for the Markov chain is determined by the following matrix equation:

$$\mathbf{\Pi}\,\mathfrak{P} = \mathbf{\Pi} \tag{5}$$

The probability distribution $\mathbf{\Pi}$ consists of $n+1$ elements, because this is the number of states of the process. Therefore, $\mathbf{\Pi} = (\Pi_1 \ \dots \ \Pi_{n+1})$.

By changing Equation (5) into the coordinate form, we have the following:

$$\begin{cases} \sum_{i=2}^{n+1} \Pi_i = \Pi_1 \\ \Pi_1 p_{1i} = \Pi_i ; \qquad i \neq 1 \\ \sum_{i=1}^{n+1} \Pi_i = 1 \end{cases} \tag{6}$$

Solving this set of equations we get

$$\Pi_1 = ½; \qquad \Pi_i = (½)p_{1i}\ ;\ i = 2, \ldots, n+1 \tag{7}$$

We can now determine the ergodic probability distribution for the semi-Markov process. Elements of it are defined by the following elements:

$$\rho_i = \frac{\Pi_i m_i}{M},\ M = \sum_{i=1}^{n+1} \Pi_i m_i \tag{8}$$

where m_i is the average time of a given state.

These parameters can be obtained from the following relationships:

$$m_i = \int_0^\infty x dQ_{1i}(x) dx\ ;\ i = 2,3,\ldots,n+1; \qquad m_1 = \sum_{i=2}^{n+1} p_{1i} m_{1i} = \sum_{i=2}^{n+1} p_{1i} \int_0^\infty x dQ_{1i}(x) dx \tag{9}$$

The steady-state availability A_s of the series system is

$$A_s = \rho_1 \tag{10}$$

The patterns that are derived concern a general case when all of the elements of the system are different. However, as a rule, all items in a series system are identical; and, for this reason,

$$\begin{gathered} Q_{21}(t) = Q_{31}(t) = \ldots = Q_{(n+1)1}(t) = G(t) \\ Q_{12}(t) = Q_{13}(t) = \ldots = Q_{1(n+1)}(t) = F(t) \end{gathered} \tag{11}$$

If $G(t)$ denotes the probability distribution of element repair time and $F(t)$ denotes the probability distribution of element work time. In further analysis this assumption will hold.

Now, we try to replace the whole series system by one *stipulated* element of reliability characteristics adequate for the system. Two probability distributions are needed: the probability distribution $F_s(t)$ of the work time of the system, and the probability distribution $G_s(t)$ of its repair time.

Due to the assumption that all elements connected in series are identical, the probability distribution of repair time of the system is identical to the probability distribution repair time of an element of the system, i.e.

$$G(t) \equiv G_s(t). \tag{12}$$

Unfortunately, with the second probability distribution is not so simple. Having the steady-state availability A_s of the system, we can calculate the average time of work $E(T_{ws})$ for the system using well-known formula

$$E(T_{ws}) = \frac{A_s}{1 - A_s} E(T_r) \tag{13}$$

where $E(T_r)$ is the average time of repair.

But that is all. In a general case, we have no possibility of getting further information on the random variable that is of interest. Nonetheless, there is an exception to this rule.

If the probability distribution of the work time of the system element is exponential, then the probability distribution of the work time of the system is also exponential and the intensity of the failures of the system is the sum of all intensities of the elements of the system. In such a case, we have complete information on the probability distribution of the work time.

If such regularity is not observed there are two possibilities. We can:

(i) use information gained from practice or
(ii) apply a simulation technique.

Neglect solution (ii). Consider the first one.

It has been observed in mining engineering that, for many pieces of equipment, the mean work time and the corresponding standard deviation remain in a certain stochastic proportion, i.e. this ratio stays approximately constant. If so, it can be presumed that the unknown standard deviation of work time is $kE(T_{ws})$ and k is a certain constant; usually $k < 1$. Thus, having information on two basic parameters of the random variable, we can presume a certain probability distribution, say the Weibull one, which will represent the probability distribution of the work time of the system. Such a Weibull distribution should have an expected value that equals $E(T_{ws})$ and the standard deviation $kE(T_{ws})$.

Hence, the following equations must hold:

$$E(T_{ws}) = \Gamma\left(1+\alpha^{-1}\right)\lambda^{-1/\alpha}$$
$$\left(kE\left(T_{ws}\right)\right)^2 = \left[\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\alpha^{-1}\right)\right] / \lambda^{\frac{2}{\alpha}} \tag{14}$$

for the probability distribution of work time of a series system given by:

$$f_s(t) = \alpha\lambda\, t^{\alpha-1}\, e^{-\lambda t^{\alpha}}, \qquad t>0,\ \alpha>0,\ \lambda>0 \tag{15}$$

Consider the reliability of a system of pair of elements.

Analysis of a symmetric pair of elements[2]

We may study an exploitation repertoire for the process of changes of states of symmetric pair. Each element can be in three states: work (W), repair (R), and standstill in reserve (S). Therefore, the set of theoretically possible states consists of $2^3 = 8$ elements; however, technically, the system can be in five states. They are as follows:

$$\{\mathfrak{S}_1, \ldots, \mathfrak{S}_5\} = \{\text{WS, WR, SW, RW, RR}\}.$$

An exploitation graph is shown in Fig. 2.

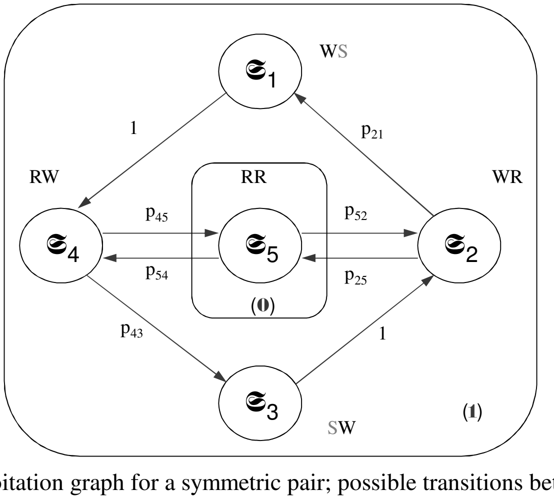

Fig. 2. Exploitation graph for a symmetric pair; possible transitions between states and corresponding probabilities
Rys. 2. Graf eksploatacyjny pary symetrycznej; możliwe przejścia pomiędzy stanami i odpowiadające im prawdopodobieństwa

[2] A lecture on a pair of elements system that has the process of changes of states following the semi-Markov scheme was given during the XL Winter Reliability School on January 2012 [6]. However, considerations were orientated on only two elements.

The passage probabilities can be calculated from following patterns:

$$
\begin{aligned}
p_{43} = 1 - p_{45} &= \int_0^\infty [1 - Q_{45}(t)]q_{43}(t)dt = \int_0^\infty [1 - F_s(t)]g_s(t)dt \\
p_{21} = 1 - p_{25} &= \int_0^\infty [1 - Q_{25}(t)]q_{21}(t)dt = \int_0^\infty [1 - F_s(t)]g_s(t)dt \\
p_{52} = 1 - p_{54} &= \int_0^\infty [1 - Q_{54}(t)]q_{52}(t)dt = \int_0^\infty [1 - G_s(t)]g_s(t)dt
\end{aligned}
\tag{16}
$$

The semi-Markov kernel $\mathfrak{Q}(t)$ – the matrix of transition between states is determined as

$$
\mathfrak{Q}(t) = \begin{pmatrix}
0 & 0 & 0 & F_s(t) & 0 \\
p_{21}G_s(t) & 0 & 0 & 0 & p_{25}F_s(t) \\
0 & F_s(t) & 0 & 0 & 0 \\
0 & 0 & p_{43}G_s(t) & 0 & p_{45}F_s(t) \\
0 & p_{52}G_s(t) & 0 & p_{54}G_s(t) & 0
\end{pmatrix}
\tag{17}
$$

The embedded Markov chain for the semi-Markov process of the symmetric pair system is

$$
\mathfrak{P} = \begin{pmatrix} \mathrm{P}_{\mathbf{11}} & \mathrm{P}_{\mathbf{10}} \\ \mathrm{P}_{\mathbf{01}} & \mathrm{P}_{\mathbf{0}} \end{pmatrix} = \left(\begin{array}{cccc|c}
0 & 0 & 0 & 1 & 0 \\
p_{21} & 0 & 0 & 0 & p_{25} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & p_{43} & 0 & p_{45} \\
\hline
0 & p_{52} & 0 & p_{54} & 0
\end{array}\right)
\tag{18}
$$

The ergodic probability distribution for the Markov chain can be obtained by solving Equation (5), keeping in mind that the matrix $\mathbf{\Pi} = (\Pi_1 \ \ldots \ \Pi_5)$ and that the sum of all probabilities obviously equals zero.

Now we can create the formula for the expected values for all five states:

$$
\begin{aligned}
m_1 &= m_{14} = \int_0^\infty t dQ_{14}(t)dt = \int_0^\infty tf_s(t)dt \\
m_2 &= p_{21}m_{21} + p_{25}m_{25} = p_{21}\int_0^\infty t dQ_{21}(t)dt + p_{25}\int_0^\infty t dQ_{25}(t)dt = \\
&= p_{21}\int_0^\infty tg_s(t)dt + p_{25}\int_0^\infty tf_s(t)dt \\
m_3 &= m_{32} = \int_0^\infty t dQ_{32}(t)dt = \int_0^\infty tf_s(t)dt \\
m_4 &= p_{43}m_{43} + p_{45}m_{45} = p_{43}\int_0^\infty t dQ_{43}(t)dt + p_{45}\int_0^\infty t dQ_{45}(t)dt = \\
&= p_{43}\int_0^\infty tg_s(t)dt + p_{45}\int_0^\infty tf_s(t)dt \\
m_5 &= p_{52}m_{52} + p_{54}m_{54} = p_{52}\int_0^\infty t dQ_{52}(t)dt + p_{54}\int_0^\infty t dQ_{54}(t)dt = \\
&= p_{52}\int_0^\infty tg_s(t)dt + p_{54}\int_0^\infty tg_s(t)dt = (p_{52} + p_{54})\int_0^\infty tg_s(t)dt
\end{aligned}
\tag{19}
$$

Thus, the ergodic probability distribution for the semi-Markov process consists of the following five elements:

$$
\rho_i = \frac{\Pi_i m_i}{M} \quad i=1,2,\dots,5 \qquad M = \sum_{i=1}^{5} \Pi_i m_i \tag{20}
$$

The steady-state availability of the multi-element symmetric pair is given by the following formula:

$$
A = \sum_{i=1}^{4} \rho_i = \frac{1}{M}\sum_{i=1}^{4} \Pi_i m_i \tag{21}
$$

Final remarks

The presented considerations and modelling approach allows the most important parameter of the system reliability to be obtained, that is, the steady--state availability. At the same time, the problems and questions related to the functioning and operation from a multi-element system reliability point of view are raised in response to both, the first work of 2011, as well as in the present work.

References

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Graniczny współczynnik gotowości wieloelementowej pary symetrycznej

Streszczenie

Artykuł jest kontynuacją rozważań zawartych w pracy pod tytułem: „Proces semi-Markowa dla pary elementów" opublikowanej w 2011 roku. W tym artykule są omawiane zagadnienia dotyczące wieloelementowej pary symetrycznej, a nie tylko samej pary elementów. Zostały zidentyfikowane i omówione problemy związane z funkcjonowaniem tego typu systemu. Dokonano analizy systemu wieloelementowego wraz z propozycją metody modelowania jego działania. Przedstawione metody uwzględniają różne przypadki w zależności od wyniku badań empirycznych; od rodzaju rozkładu prawdopodobieństwa czasów stanów elementów, z których składa się system. Zaprezentowana metoda modelowania dostarcza rozwiązań, które pozwalają na uzyskanie podstawowych parametrów niezawodnościowych rozważanego systemu wieloelementowego.