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The outline of the assessment of durability against surface wear of a construction element with the use of the distribution of time of the exceedence of limit state (admissible state)

Key words

Reliability, durability, wear, admissible state (limit state).

Słowa kluczowe

Niezawodność, trwałość, zużycie, stan dopuszczalny (graniczny).

Summary

There are two trends concerning the examination of the wear processes of construction elements in the process of operation. The first trend concerns the physics of wear process, and the second trend concerns the construction of mathematical models describing the process of the increase in wear results. This article concerns the second trend and includes a model describing an increase in wear results until reaching limit states.

The model includes the distribution of time of the exceedance of an accessible state (limit state). After transforming this model, calculations of durability will be reduced to the use of the standardised normal distribution.

In conclusion, numerical examples including the assessment of the durability of an aircraft tire are presented.

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1. Introduction

There are two trends concerning the examination of the wear processes of construction elements in the aeronautical engineering:

- The first one concerns the interpretation of the physics of wear process.
- The second one concerns the construction of mathematical models describing wear processes.

This article concerns the second trend.

The process of the surface wear of mechanical elements usually involves a decrease in their mass and change of their size. The change of an element's size leads to a growth in clearance that causes an increase in vibration and a stoppage rate in kinematic systems, which results in the loss of airworthiness.

Let us assume that we are examining a wear process of a construction element as it is shown in Fig.1.

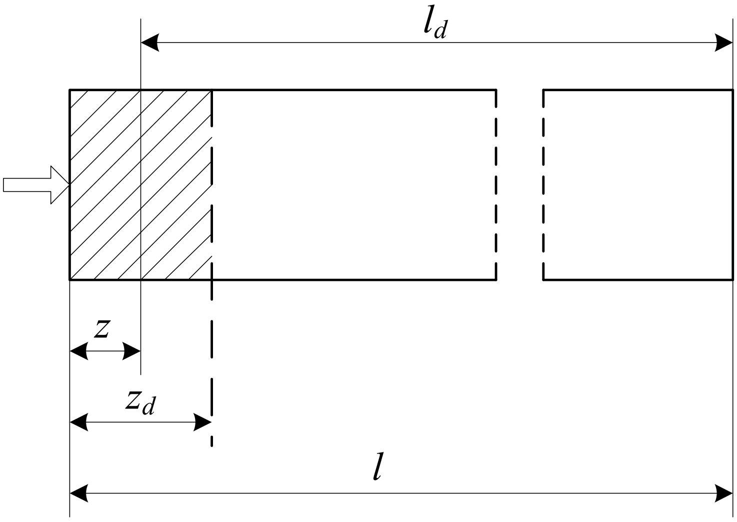

Fig. 1. A diagram of an element with a marked wear area: l – linear dimension of an element, l_d – a parameter describing a state of an element, z – a current wear of an element, z_d – admissible wear of an element

Rys. 1. Schemat elementu z zaznaczonym obszarem zużycia: l – wymiar liniowy elementu, l_d – parametr opisujący stan elementu, z – zużycie bieżące elementu, z_d – zużycie dopuszczalne elementu

A current wear is marked with "z". When $z < z_d$ then an element is in an operational state, when $z \geq z_d$ an element loses its operational state.

The process of the surface wear of an element is of random character because of a specific character of load and an influence of destructive process. Let us consider the wear process of a construction element as a random process occurring during the operation of an aircraft.

2. Determining the density function describing the growth of an element's wear

The following assumptions are accepted:

1. A technical condition of an element is determined with one diagnostic parameter, which is marked with "z".
2. Value of parameter "z" changes only during device operation, i.e. during flight.
3. Parameter "z" is non-decreasing.
4. A change in diagnostic parameter "z" is determined by the following dependence:

$$\frac{dz}{dN} = C \tag{1}$$

where: C – a random variable which depends on the operation conditions of an element;
N – a number of flights of an aircraft.

5. If $z \in [0, z_d]$, then an element is in an operational state, otherwise it will be in a non-operational state.
6. Flight intensity is determined by the following dependency:

$$\lambda = \frac{P}{\Delta t} \tag{2}$$

where: Δt – time interval in which flight can be performed with probability P,
P – probability of flight in time interval of length Δt.

The time interval of length Δt shall be matched with flight intensity in order to perform inequality:

$$\lambda \Delta t \leq 1 \tag{3}$$

Using flight intensity λ, we can determine the number of flights performed until moment t :

$$N = \lambda t \tag{4}$$

Using dependency (4), we can write down formula (1) in the following form:

$$\frac{dz}{dt} = \lambda c \tag{5}$$

The dynamics of diagnostic parameter changes will be described with the following difference equation:

$$U_{z,t+\Delta t} = (1-\lambda\Delta t)U_{z,t} + \lambda\Delta t\, U_{z-\Delta z,t} \tag{6}$$

where: $U_{z,t}$ – probability that in moment t the diagnostic parameter value will be z;
Δz – increase of diagnostic parameter z during one flight of an aircraft.

In functional notation equation (6) has the following form:

$$u(z,\, t+\Delta t) = (1-\lambda\Delta t)\, u(z,t) + \lambda\Delta t\, u(z-\Delta z, t) \tag{7}$$

where: $u(z,t)$ – the density function of the probability of a value of diagnostic parameter z in moment t;
$(1-\lambda\Delta t)$ – probability that there will be no flight in time interval of length Δt;
$\lambda\Delta t$ – probability of flight in time interval of length Δt.

After transformations of difference equation (7), we obtain the following partial differential equation [2]:

$$\frac{\partial u(z,\, t)}{\partial z} = -\lambda\,\Delta z\,\frac{\partial u(z,\, t)}{\partial z} + \frac{1}{2}\lambda(\Delta z)^2\frac{\partial^2 u(z,\, t)}{\partial z^2} \tag{8}$$

where: $\Delta z = c$.

Because c is a random variable, we derive the mean value:

$$E[c] = \int_{c_d}^{c_g} c\, f(c)\, dc \tag{9}$$

where: $f(c)$ – the density function of random variable c;
c_g, c_u – limits of inconstancy of value c.

Taking dependency (9) into consideration, we can write down differential equation (8) in the following form:

$$\frac{\partial u(z,\, t)}{\partial t}\Delta t = -\lambda\, E[c]\,\frac{\partial u(z,\, t)}{\partial z} + \frac{1}{2}\lambda(E[c])^{2}\ \frac{\partial^{2} u(z,\, t)}{\partial z^{2}} \tag{10}$$

where: $\lambda\, E[c]$ – average increase of parameter value per unit of time;

$\lambda(E[c])^2$ – average square of the increase of the diagnostic parameter value per unit of time.

The solution of equation (10) is the unknown density function of the probability of a random variable z :

$$u(z,t) = \frac{1}{\sqrt{2\pi\, A(t)}}\ e^{-\frac{(z-B(t))^2}{2A(t)}} \tag{11}$$

where:

$$B(t) = \int_0^t \lambda\, E[c]\, dt = \lambda\, E[c]\, t\,,$$

$$A(t) = \int_0^t \lambda\, (E[c]\,)^2\, dt = \lambda\, E[c]^2\, t\,.$$

Assuming that:

$$b = \lambda\, E[c]\ ,$$

$$a = \lambda\, E[c]^{\,2}\,,$$

the density function (11) has the following form:

$$u(z,t) = \frac{1}{\sqrt{2\pi\, a\, t}}\ e^{-\frac{(z-bt)^2}{2at}} \tag{12}$$

Dependency (12) is a probabilistic characteristic of an increase in wear as a function of flying time of an aircraft.

3. Distribution of time (flying time) of the exceedance of an admissible state (limit state) of "z" parameter

The probability of the exceedance of an admissible value by a current value of diagnostic parameter z can be written down in the following form [5]:

$$Q(t; z_d) = \int_{z_d}^{\infty} \frac{1}{\sqrt{2\pi\, at}}\, e^{-\frac{(z-bt)^2}{2at}}\, dz \tag{14}$$

The density function of the distribution of the time of the exceedance of an admissible state z_d has the following form:

$$f(t) = \frac{\partial}{\partial t} Q(t; z_d) \tag{15}$$

Hence,

$$f(t) = \frac{\partial}{\partial t} \int_{z_d}^{\infty} \frac{1}{\sqrt{2\pi\, at}}\, e^{-\frac{(z-bt)^2}{2at}}\, dz \tag{16}$$

$$f(t) = \int_{z_d}^{\infty} \left\{ \frac{\partial}{\partial t} \left[\frac{1}{\sqrt{2\pi\, at}}\, e^{-\frac{(z-bt)^2}{2at}} \right] \right\} dz \tag{17}$$

After calculating the derivation, we obtain:

$$f(t)_{z_d} = \int_{z_d}^{\infty} \left[u(z,t) \left(\frac{z^2 - b^2 t^2 - at}{2\, at^2} \right) \right] dz \tag{18}$$

The primitive function toward the integrand of dependency (18) has the following form:

$$w(z,t) = u(z,t) \left(-\frac{z + bt}{2t} \right) \tag{19}$$

Let us calculate integral (18):

$$f(t)_{z_d} = u(z,t)\left(-\frac{z+bt}{2t}\right)\Bigg|_{z_d}^{\infty} = \frac{z_d + bt}{2t}\,\frac{1}{\sqrt{2\pi\, at}}\; e^{-\frac{(z_d - bt)^2}{2at}} \qquad (20)$$

Dependency (20) determines the density function of the time of the first exceedance of a current value of parameter *z* beyond an admissible state.

4. Reliability and durability of a construction element assessed with respect to diagnostic parameter *z*

The formula describing the reliability of an element has the following form:

$$R(t) = 1 - \int_0^t f(t)_{z_d}\, dt \qquad (21)$$

where: density function $f(t)_{z_d}$ determined by formula (20).

The unreliability of an element can be determined from dependency (22):

$$Q(t) = \int_0^t \frac{z_d + bt}{2t} \cdot \frac{1}{\sqrt{2\pi\, at}}\; e^{-\frac{(z-bt)^2}{2at}}\, dt \qquad (22)$$

Integral (22) shall be reduced. It is seen that integrand can be written in the following form:

$$\frac{z_d + bt}{2t} \cdot \frac{1}{\sqrt{2\pi\, at}}\; e^{-\frac{(z_d - bt)^2}{2at}} = \frac{z_d + bt}{2t} \cdot \frac{1}{\sqrt{2\pi\, at}}\; e^{-\frac{(bt - z_d)^2}{2at}},$$

and the problem is reduced to solving indefinite integral

$$\int \frac{(z_d + bt)}{2t} \cdot \frac{1}{\sqrt{2\pi\, at}}\; e^{-\frac{(bt - z_d)^2}{2at}}\, dt \qquad (23)$$

We make a substitution in the above-mentioned integral [1]:

$$\frac{(bt - z_d)^2}{2at} = u\,.$$

Hence,

$$\frac{du}{dt} = \frac{bt + z_d}{2at^2}\left(bt - z_d\right),$$

$$dt = \frac{2at^2}{\left(bt + z_d\right)\left(bt - z_d\right)}\ du\ .$$

After substitution, integral (23) has the following form:

$$\int \frac{z_d + bt}{2t} \cdot \frac{1}{\sqrt{2\pi\, at}}\, e^{-u} \cdot \frac{2at^2}{(bt + z_d)(bt - z_d)}\ du = \frac{1}{2\sqrt{\pi}} \int \frac{1}{\sqrt{u}}\, e^{-u} du \qquad (24)$$

Then, we make the second substitution:

$$\sqrt{u} = w,$$

$$\frac{dw}{du} = \frac{1}{2\sqrt{u}}\ ,$$

$$\frac{du}{dw} = 2w\ ,$$

$$du = 2w\,dw\ .$$

Taking the above-mentioned dependencies into consideration, integral (24) can be written in the following form:

$$\frac{1}{2\sqrt{\pi}} \int \frac{1}{w}\, e^{-w^2}\ 2w\,dw = \frac{1}{\sqrt{\pi}} \int e^{-w^2} dw \qquad (25)$$

We make the next substitution:

$$w^2 = \frac{y^2}{2},$$

$$2w\,dw = y\ dy\ ,$$

$$dw = \frac{y}{2w}\, dy\ ,$$

$$dw = \frac{y}{\sqrt{2}}\ .$$

Hence, we obtain an integral in the following form:

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{y^2}{2}}\, dy \tag{26}$$

where:

$$y = \frac{bt - z_d}{\sqrt{at}}\ .$$

Substituting the above-obtained results into formula (21) and remembering about an appropriate notation of integration limits, we obtain a formula describing reliability:

$$R(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{bt - z_d}{\sqrt{at}}} e^{-\frac{y^2}{2}}\, dy \tag{27}$$

The distribution function of the standardised normal distribution has the following form:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy\ .$$

The final formula determining the reliability of a construction element has the following form:

$$R^*(t) = 1 - \Phi\left(\frac{b^* t - z_d}{\sqrt{a^* t}}\right) \tag{28}$$

where b^* and a^* are coefficients after estimation on the basis of data obtained from operation of the aircraft.

Thus, damage risk of an element can be determined from the dependency below:

$$Q^* = 1 - R^*(t) = \Phi(\gamma) \tag{29}$$

where:

$$\gamma = \frac{bt - z_d}{\sqrt{a^* t}} \tag{30}$$

Assuming a particular damage risk level, we find γ (by reading tables of standardised normal distribution). Knowing value γ, we can determine durability (i.e. *t*) from dependency (30). For this purpose, dependency (30) was transformed into the following quadratic equation:

$$b^{*2}t^2 - \left(\gamma^2 a^* + 2b^* z_d\right)t + z_d^2 = 0\,.$$

Hence, durability:

$$T = \frac{\left(\gamma^2 a^* + 2b^* z_d\right) - \sqrt{\left(2b^* z_d + \gamma^* a^*\right)^2 - 4b^{*2} z_d^2}}{2b^{*2}} \tag{31}$$

5. Numerical examples and final notes

The example described below includes an aircraft tire. The basic tire dimensions, which are important while assessing its wear, are consistent with Fig.1. An aircraft tire loses its operational state when tread wear exceeds 7mm. So, the task includes the assessment of the number of landings, i.e. the number of flights which can be performed before $z \geq z_d$.

The density function (12), which depends on flying time, shall be described as dependent on the number of flights. Thus, dependency (12) has the following form:

$$u(z, N) = \frac{1}{\sqrt{2\pi\,\hat{a}\,N}}\; e^{-\frac{\left(z - \hat{b}N\right)}{2\hat{a}N}} \tag{32}$$

where: $N = \lambda t$ – the number of flights;

$\hat{a} = E[c]^2$ – mean square of wear per flight;

$\hat{b} = E[c]^2$ – mean wear per flight.

In order to use density function (32), we must assess $\hat{a}$ and $\hat{b}$ using data on tire wear.

Table 1. The obtained data on tire wear in the function of flight numbers
Tabela 1. Uzyskane dane zużycia opon w zależnośći od liczby lotów

Number of measurements r	1	2	3	4	5	6	7
Number of landings n	5	10	15	20	30	40	50
Wear z_n [mm]	0.105	0.1350	0.210	0.325	0.470	0.730	0.830

A plausibility function was used to determine $\hat{b}$ and $\hat{a}$. Hence, calculation formulas have the following forms:

$$\hat{b}^* = \frac{z_n}{n} \tag{33}$$

$$\hat{a}^* = \frac{1}{r}\sum_{k=0}^{n-1} \frac{\left[z_{k+1} - z_k - \hat{b}^*(n_{k+1} - n_k)\right]^2}{n_{k+1} - n_k} \tag{34}$$

Using the above-presented formulas and data in the Table 1, we can determine values $\hat{b}^*$ and $\hat{a}^*$:

$$\hat{b}^* = \frac{z_{50}}{50} = \frac{0{,}830}{50} = 0{,}0166,$$

$$\hat{a}^* = 0{,}00051.$$

The risk of exceedance of admissible wear was z_d = 7 mm close to zero. Hence,

$$1 - R^*(t) = \Phi(\gamma) \approx 0.$$

Thus, it can be written:

$$-3{,}0 = \frac{\hat{b}^* N - z_d}{\sqrt{\hat{a}^* N}} \tag{35}$$

$$9\hat{a}^{*}N = \left(\hat{b}^{*}\,N - z_d\right)^2,$$

$$9\,\hat{a}^{*}N = \hat{b}^{*2}\,N^2 - 2\,z_d\hat{b}^{*2}\,N + z_d^2,$$

$$\hat{b}^{*2}\,N^2 - \left(9\hat{a}^{*} - 2\,z_d\hat{b}^{*}\right)N + z_d^2 = 0.$$

After transformations, the formula determining tire durability has the following form:

$$N^{*} = \frac{\left(9\,\hat{a}^{*} + 2z_d\hat{b}^{*}\right) - \sqrt{\left(9\,\hat{a}^{*} + 2z_d\hat{b}^{*}\right)^2 - 4\hat{b}^{*2}z_d^2}}{2\hat{b}^{*2}}.$$

After substituting the determined values into the formula presented above, the following formula was obtained:

$$N^{*} = \frac{(9\cdot 0{,}00051 + 2\cdot 7\cdot 0{,}0166) - \sqrt{(9\cdot 0{,}00051 + 2\cdot 7\cdot 0{,}0166)^2 - 4\cdot 0{,}0166^2\,7^2}}{2\cdot 0{,}0166^2} =$$

$$= \frac{0{,}232859 - 0{,}0112827}{0{,}000552} = \frac{0{,}221576}{0{,}000552} \cong 400{,}0.$$

The accuracy of wear assessment depends on the accuracy of numerical data on wear.

In conclusion, we will obtain the same results if we reduce our analyses to distribution (12) determining wear as a function of flying time and reliability assessment:

$$R(t) = \int_{-\infty}^{z_d} u(z,t)\,dz,$$

where: $u(z, t)$ – density function of wear determined by dependency (12).

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Zarys metody oceny trwałości na zużycie powierzchniowe elementu konstrukcji z wykorzystaniem rozkładu czasu przekroczenia stanu granicznego (dopuszczalnego)

S t r e s z c z e n i e

W ramach problematyki dotyczącej badania procesów zużywania się elementów konstrukcji w procesie eksploatacji można wyróżnić dwa kierunki. Pierwszy dotyczący wyjaśnienia fizyki zużywania, drugi – budowy modeli matematycznych opisujących przebieg narastania skutków zużywania. Niniejszy artykuł dotyczy drugiego kierunku i obejmuje model opisujący narastanie skutków zużywania do chwili osiągania stanów granicznych.

W modelu tym wykorzystano rozkład czasu przekroczenia stanu dopuszczalnego (granicznego). W wyniku przekształceń tego modelu obliczenie trwałości sprowadza się do wykorzystania rozkładu normalnego standaryzowanego.

Na zakończenie przedstawiono przykład liczbowy, obejmujący ocenę trwałości opony lotniczej.

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