SCIENTIFIC PROBLEMS
OF MACHINES OPERATION
AND MAINTENANCE
4 (164) 2010

HENRYK TOMASZEK [*], MICHAŁ JASZTAL[**]

Outline of a method for fatigue life determination for selected aircraft's elements

K e y w o r d s

Fatigue life, Paris' equation, constant and variable amplitude loading.

S ł o w a k l u c z o w e

Trwałość zmęczeniowa, równanie Parisa, stało- oraz zmiennoamplitudowe widmo obciążenia.

S u m m a r y

This paper describes a method for the evaluation of the fatigue life of a structural component of an aircraft for constant and variable amplitude loading, using deterministic description of fatigue crack growth based on Paris equation with corrective coefficients. The coefficients take into consideration crack and element geometry and phenomena connected with variable amplitude loading effects. Final equations for fatigue life calculations were carried out for two special cases: when the exponent of the Paris formula is $m = 2$ and $m = 4$. Examples show the application of the method and indicate numerical verification of the mathematical model.

[*] Air Force Institute of Technology, Księcia Bolesława 6 Street, 01-494 Warsaw, Poland.
[**] Military University of Technology, Kaliskiego 2 Street, 00-908 Warsaw, Poland, mjasztal@wat.edu.pl, tel. +48 22 683 77 89.

1. Introduction

This paper assumes that crack l_o occurs in a given structural element during operation, and it increases under variable loading to admissible (safety) length l_d. The crack growth process, approached in a deterministic way, has been described with the Paris formula in the following form:

$$\frac{dl}{dN} = C(\Delta K)^m \tag{1}$$

where: ΔK – range of stress intensity factor,
C, m – material constants,
N – variable which represent number of load cycles.

According to equation (1), structural element fatigue life can be express by the following equation:

$$N_{l_d} = \int_{l_0}^{l_d} \frac{1}{C(\Delta K)^m} dl \tag{2}$$

The above-mentioned integral (2) is sometimes hard to solve. Troubles are caused by the coefficients, which are dependent on actual crack length. Mathematical model consists the coefficients which improve accuracy but on the other hand makes analytical solution difficult. An example of this coefficient is a variable described as M_k which specifies the influence of crack location and dimensions in relation to structural element dimensions on crack growth velocity. Admissible crack length l_d can be described with use of the stress intensity factor in the following form:

$$K = M_k \sigma \sqrt{\pi l} \tag{3}$$

Stress intensity factor (3) becomes a quantity of a critical value K_c when the crack length and the stress takes critical values l_{kr} and σ_{kr} respectively. Then it is called “resistance of the material to cracking”.

$$K_c = M_k \sigma_{kr} \sqrt{\pi l_{kr}} \tag{4}$$

Exceeding the critical value of the crack length usually leads to a catastrophic damage to the component. If a factor of safety is introduced, one can find the admissible value of the crack. The computational formula takes the following form:

$$l_d = \frac{K_c^2}{k M_k^2 \sigma_{kr}^2 \pi} \tag{5}$$

where: k – factor of safety.

2. Fatigue life of selected structural elements under constant amplitude loading

Equation (1) in the developed form is as follows:

$$\frac{dl}{dN} = UCM_k^m (\Delta\sigma)^m \pi^{\frac{m}{2}} l^{\frac{m}{2}} \tag{6}$$

where: U – empirical function of crack closure contribution to crack growth relates to stress ratio R,

$\Delta\sigma$ – the range of stress in one cycle $\Delta\sigma = \sigma_{max} - \sigma_{min}$,

M_k – geometrical coefficient (Fig. 1) defined for specified geometry by relation [1]:

$$M_k = 1 - 0{,}1\left(\frac{l}{w}\right) + \left(\frac{l}{w}\right)^2 \tag{7}$$

w – structural element dimension in the direction of crack growth.

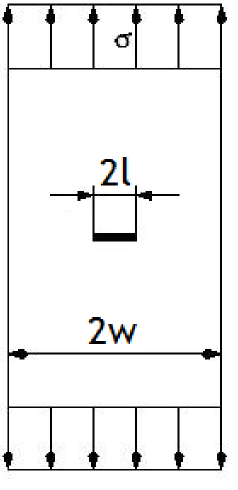

Fig. 1. Element geometry with central crack
Rys. 1. Geometria elementu z centralnym pęknięciem

Notation of the coefficient M_k was assumed in the following form:

$$M_k = a + bx + cx^2 \tag{8}$$

where: $x = l,\ a = 1,\ b = -\frac{0{,}1}{w},\ c = \frac{1}{w^2}.$

Taking the above-mentioned assumption into account, Equation (6) takes the following form:

$$\frac{dx}{dN} = UC(a + bx + cx^2)^m(\Delta\sigma)^m\pi^{\frac{m}{2}}x^{\frac{m}{2}} \tag{9}$$

The indefinite integral has the following form:

$$\int \frac{dx}{(a+bx+cx^2)^m x^{\frac{m}{2}}} = \int UC(\Delta\sigma)^m\,\pi^{\frac{m}{2}}dN \tag{10}$$

Hence, fatigue life of the structural element for assumed type of load is as follows:

$$N = \frac{1}{UC(\Delta\sigma)^m\pi^{\frac{m}{2}}}\int_{l_0}^{l_d}\frac{dx}{x^{\frac{m}{2}}(a + bx + cx^2)^m}$$

For $R = a + bx + cx^2$ the equation has the form:

$$N = \frac{1}{UC(\Delta\sigma)^m\pi^{\frac{m}{2}}}\int_{l_0}^{l_d}\frac{dx}{x^{\frac{m}{2}}(R)^m} \tag{11}$$

Calculation of Integral (11) in analytical form is difficult. Hence, the fatigue life description was carried out for two special cases: when the exponent of the Paris formula is $m = 2$ and $m = 4$.

Fatigue life determination for $m = 2$

Formula (11) for $m = 2$ takes the following form:

$$N = \frac{1}{UC(\Delta\sigma)^2\pi}\int_{l_0}^{l_d}\frac{dx}{xR^2} \tag{12}$$

where: $R = (a + bx + cx^2)$.

Indefinite integral [2]:

$$\int \frac{dx}{xR^2} = \frac{1}{2a^2} ln \frac{x^2}{R} + \frac{1}{2aR}\left(1 - \frac{b(b+2cx)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right) \int \frac{dx}{R} \quad (13)$$

Integral $\int \frac{dx}{R}$ has following solution [2]:

$$\int \frac{dx}{R} = \begin{cases} \frac{-2}{\sqrt{-\Delta}} arctgh \frac{b+2cx}{\sqrt{-\Delta}} & \text{dla } \Delta < 0 \\ \frac{-2}{b+2cx} & \text{dla } \Delta = 0 \\ \frac{2}{\sqrt{\Delta}} arctg \frac{b+2cx}{\sqrt{\Delta}} & \text{dla } \Delta > 0 \end{cases} \quad (14)$$

where: $\Delta = 4ac - b^2$.

For the assumption that $\Delta > 0$ Indefinite Integral (13) takes the form:

$$\int \frac{dx}{xR^2} = \frac{1}{2a^2} ln \frac{x^2}{R} + \frac{1}{2aR}\left(1 - \frac{b(b+2cx)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right) \frac{2}{\sqrt{\Delta}} arctg \frac{b+2cx}{\sqrt{\Delta}} \quad (15)$$

Calculating the definite integral:

$$\int_{l_0}^{l_d} \frac{dx}{xR^2} = \left[\frac{1}{2a^2} ln \frac{{l_d}^2}{R(l_d)} + \frac{1}{2aR(l_d)}\left(1 - \frac{b(b+2cl_d)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right) \frac{2}{\sqrt{\Delta}} arctg \frac{b+2cl_d}{\sqrt{\Delta}}\right] -$$

$$\left[\frac{1}{2a^2} ln \frac{{l_0}^2}{R(l_0)} + \frac{1}{2aR(l_0)}\left(1 - \frac{b(b+2cl_0)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right) \frac{2}{\sqrt{\Delta}} arctg \frac{b+2cl_0}{\sqrt{\Delta}}\right] \quad (16)$$

where: $R(l_d)=(a + bl_d + c(l_d)^2)$; $R(l_0)=(a + bl_0 + c(l_0)^2)$

The fatigue life of the element for $m = 2$ can be obtained from the following:

$$N = \frac{1}{UC(\Delta\sigma)^2\pi} \int_{l_0}^{l_d} \frac{dx}{xR^2} \quad (17)$$

where: integral $\int_{l_0}^{l_d} \frac{dx}{xR^2}$ is described by Equation (16).

Fatigue life determination for *m* = 4
The main task in this case is to the calculation of indefinite integral:

$$\int \frac{dx}{R^4 x^2}$$

Calculating Integral (18):

$$\int \frac{dx}{R^4x^2} = \frac{1}{3a^5}\Bigg(-\frac{3a}{x} + \frac{a^3(b^3-3abc+b^2cx-2ac^2x)}{(-b^2+4ac)(a+x(b+cx))^3} -$$

$$\frac{a^2(3b^5-22ab^3c+35a^2bc^2+3b^4cx-20ab^2c^2x+22a^2c^3x)}{(b^2-4ac)^2(a+x(b+cx))^2} +$$

$$\left(\frac{3a(-3b^7+34ab^5c-124a^2b^3c^2+134a^3bc^3-3b^6cx+32ab^4c^2x-104a^2b^2c^3x+76a^3c^4x)}{(b^2-4ac)^3(a+x(b+cx))}\right) -$$

$$\frac{12(b^8-14ab^6c+70a^2b^4c^2-140a^3b^2c^3+70a^4c^4)arctg\left[\frac{b+2cx}{\sqrt{-b^2+4ac}}\right]}{(-b^2+4ac)^{7/2}} - 12b\cdot log(x) + 6b\cdot$$

$$log\big(a+x(b+cx)\big)\Bigg) \quad (18)$$

Finally, applying a similar procedure like for $m = 2$ one can determine the fatigue life of the structural element for the Paris formula exponent $m = 4$.

3. Fatigue life of selected structural elements under variable amplitude loading

In this model, further assumptions are as follows:

1. Aircraft structural element works during operation under variable loading.
2. Fatigue loading of the component is determined with some spectrum of loads, set up using a pattern of loading in the standard flight of aircraft.
3. Spectrum of loads is determined as follows:
   - The load spectrum consist of N_c cycles.
   - Load cycles can be ordered in L stress levels and each level has maximum stress values $\sigma_1^{max}, \sigma_2^{max}, \ldots, \sigma_L^{max}$.
4. The number of maximum stress values repetitions in the load spectrum is as follows:
   σ_1^{max} occurs n_1 times, σ_2^{max} occurs n_2 times, …, σ_L^{max} occurs n_L times.

The number of specified load level repetitions during a standard flight of the aircraft is as follows: $N_c = \sum_{i=1}^{L} n_i$.

5. The minimum stress value at the specified load levels is described by the following formula:

$$\sigma_{i,śr}^{min} = \frac{\sigma_{i,1}^{min} + \sigma_{i,2}^{min} + \dots + \sigma_{i,n_i}^{min}}{n_i}, \text{ where } i = 1,2,\dots,n_i.$$

6. Table 1 shows the maximum and minimum stress levels and the frequency of stress levels appearing in the spectrum:

Table 1. Maximum σ_i^{max} and minimum $\sigma_{i,śr}^{min}$ stress values in the cycles, and frequencies of their appearing in the spectrum P_i

Tabela 1. Zestawienie maksymalnych σ_i^{max} and minimum $\sigma_{i,śr}^{min}$ wartości naprężeń w cyklach oraz częstości ich występowania P_i

σ_i^{max}	σ_1^{max}	σ_2^{max}	…	σ_i^{max}	…	σ_L^{max}
$\sigma_{i,śr}^{min}$	$\sigma_{1,śr}^{min}$	$\sigma_{2,śr}^{min}$	…	$\sigma_{i,śr}^{min}$	…	$\sigma_{L,śr}^{min}$
P_i	$P_1 = \frac{n_1}{N_c}$	$P_2 = \frac{n_2}{N_c}$	…	$P_i = \frac{n_i}{N_c}$	…	$P_L = \frac{n_L}{N_c}$

7. Table 2 presents stress ratio coefficients and empirical function U of the crack closure contribution to crack growth as related to stress ratio $\hat{R}_i$:

Table 2. Stress ratios $\hat{R}_i$ and empirical coefficients of influence on crack growth U_i

Tabela 2. Zestawienie współczynników asymetrii cyklu oraz współczynników uwzględniających ich wpływ na prędkość pękania

$cykle\ i$	1	2	…	i	…	L
$\hat{R}_i$	$\hat{R}_1$	$\hat{R}_2$	…	$\hat{R}_i$	…	$\hat{R}_L$
U_i	U_1	U_2	…	U_i	…	U_L

where: $\hat{R}_i = \frac{\sigma_{i,śr}^{min}}{\sigma_i^{max}}$, $U_i = \propto_1 + \propto_2 \hat{R}_i + \propto_3 \hat{R}_i^{\,2}$; $\propto_1, \propto_2, \propto_3$ – empirical coefficients.

8. Table 3 consist of stress range levels:

$$\Delta\sigma_i = \sigma_i^{max} - \sigma_{i,śr}^{min}$$

Table 3. Range of stress $\Delta\sigma_i$ and frequencies of their appearing in the spectrum P_i
Tabela 3. Zestawienie wartości zakresu zmian naprężeń $\Delta\sigma_i$ oraz częstości ich występowania P_i

$cycle\ types$	1	2	…	i	…	L
$\Delta\sigma_i$	$\Delta\sigma_1$	$\Delta\sigma_2$	…	$\Delta\sigma_i$	…	$\Delta\sigma_L$
P_i	P_1	P_2	…	P_i	…	P_L

9. Table 4 consist of retardation coefficients C_i^P for specified levels which takes into consideration the influence of overload cycles on crack growth rate:

$$\Delta\sigma_{i,ef} = C_L^P \Delta\sigma_i$$

where: C_i^P – retardation coefficients.

Table 4. Range of effective stress $\Delta\sigma_{i,ef}$ which takes into consideration effect of overload cycles
Tabela 4. Zestawienie wartości zakresu zmian naprężeń efektywnych $\Delta\sigma_{i,ef}$ uwzględniających występowanie cykli przeciążających

$cycle\ types$	1	2	…	i	…	L
$coefficients$	C_1^P	C_2^P	…	C_i^P	…	C_L^P
$\Delta\sigma_{i,ef}$	$\Delta\sigma_{1,ef}$	$\Delta\sigma_{2,ef}$	…	$\Delta\sigma_{i,ef}$	…	$\Delta\sigma_{L,ef}$

The crack growth process, approached in a deterministic way, has been described with the Paris formula in (1). For the above-mentioned assumptions, Formula (1) for *i*-th type of load cycles has following form:

$$\frac{dl}{dN} = CU_i M_k^m \left(\Delta\sigma_{i,ef}\right)^m \pi^{\frac{m}{2}} l^{\frac{m}{2}} \tag{20}$$

Taking into consideration all types of load cycles, Formula (20) has the following form:

$$\frac{dl}{dN} = C\pi^{\frac{m}{2}} \left(\sum_{i=1}^{L} P_i\, U_i \left(\Delta\sigma_{i,ef}\right)^m\right) M_k^m l^{\frac{m}{2}} \tag{21}$$

Assuming that

$$l = x;\ M_k = R;\ R = (a + bx + cx^2)$$

Then

$$\frac{dx}{dN} = C\pi^{\frac{m}{2}} \left(\sum_{i=1}^{L} P_i\, U_i \left(\Delta\sigma_{i,ef}\right)^m\right) (a + bx + cx^2)^m x^{\frac{m}{2}} \tag{22}$$

The integration of equation (22) produces the following fatigue life formula:

$$N = \frac{1}{C\pi^{\frac{m}{2}}\left(\sum_{i=1}^{L} P_i U_i \left(\Delta\sigma_{i,ef}\right)^m\right)} \int_{l_0}^{l_d} \frac{dx}{x^{\frac{m}{2}} R^m} \tag{23}$$

Equation (23) provides the possibility of the fatigue life calculation for structural elements with use of a variable amplitude loading description, which take into consideration *L* load types.

Using these results one can derive a formula for the fatigue life calculation of a structural element for the Paris formula exponent $m = 2$ or $m = 4$.

4. Final remarks and a computational example

The method was verified by predicting the crack behavior and fatigue life estimation for steel sheet subjected to variable amplitude load program. The specimens geometry given in Fig. 2 were cut out from the sheet. Subsequently, a through-thickness central hole of 5 mm in diameter, was cut inside each specimen. The central hole had on each side a through-thickness saw cut of 2.5 mm length and an initial pre-crack of 2.5 mm length, the total length of the initial crack was equaled to $2l = 20$ mm. The hole served as crack initiator.

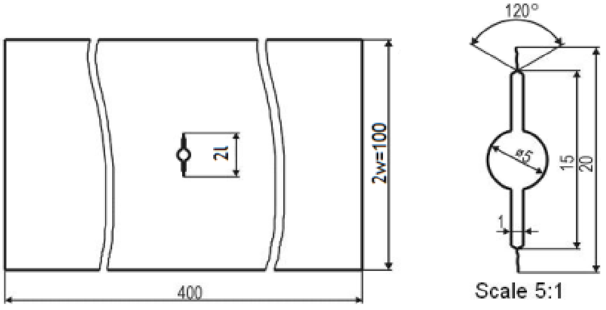

Fig. 2. Geometry of specimen
Rys. 2. Geometria próbki

The calculation was carried out for a variable amplitude load spectrum. The characteristic of the spectrum is presented in Table 5. The table contains maximum stress σ_i^{max}, average minimum stress $\sigma_{i,śr}^{min}$ and effective stress range values $\Delta\sigma_{i,ef}$ for the established 7 load levels with frequency of stress levels appearing in the spectrum. Furthermore, the table contains stress ratio coefficients $\hat{R}_i$ and empirical function U_i of crack closure contribution to crack growth relates to stress ratio:

$$U_i = 0,55 + 0,33\hat{R}_i + 0,12\hat{R}_i^{\,2}$$

The calculation was carried out with use of following material constants:

$$m = 2$$

$$C = 8 \cdot 10^{-10}.$$

Table 5. Quantities which describe loading spectrum
Tabela 5. Wielkości charakteryzujące widmo obciążeń

Load level i	1	2	3	4	5	6	7
Number of cycles	1	5	4	10	30	50	140
σ_i^{max} [MPa]	186	159	141	129	112	93	72
$\sigma_{i,śr}^{min}$ [MPa]	-28	-13	8	17	23	27	27
Coefficient $\hat{R}_i$	-0.1505	-0.0818	0.0567	0.1317	0.2053	0.2903	0.375
Stress range $\Delta\sigma_{i,ef}$ [MPa]	214	172	133	112	89	66	45
Empirical function U_i	0.5030	0.5238	0.5691	0.5955	0.6228	0.6559	0.6906
Load level contribution into load spectrum P_i	0.0042	0.0208	0.0167	0.0417	0.125	0.2083	0.5833

Furthermore, the calculation of fatigue life was carried out for crack growth from initial crack length l_0 = 10 mm to admissible crack length l_d = 27 mm. The admissible crack length was calculated using Formula (5). Retardation coefficient C_i^P which takes into consideration the influence of overload cycles on crack growth rate was assumed as $C_i^P = 1$. Finally, Formula (23) was used for the calculation fatigue life for m = 2:

$$N_{l_d} = \frac{1}{C\pi\left(\sum_{i=1}^{L} P_i\, U_i\left(\Delta\sigma_{i,ef}\right)^2\right)} \int_{l_0}^{l_d} \frac{dx}{xR^2}$$

Integral described by Formula (16):

$$\int_{l_0}^{l_d} \frac{dx}{xR^2} = \left[\frac{1}{2a^2} ln \frac{{l_d}^2}{R(l_d)} + \frac{1}{2aR(l_d)}\left(1 - \frac{b(b+2cl_d)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right)\frac{2}{\sqrt{\Delta}} arctg \frac{b+2cl_d}{\sqrt{\Delta}}\right] - \left[\frac{1}{2a^2} ln \frac{{l_0}^2}{R(l_0)} + \frac{1}{2aR(l_0)}\left(1 - \frac{b(b+2cl_0)}{\Delta}\right) - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right)\frac{2}{\sqrt{\Delta}} arctg \frac{b+2cl_0}{\sqrt{\Delta}}\right]$$

where: $R(l_d)=(a + bl_d + c(l_d)^2)$; $R(l_0)=(a + bl_0 + c(l_0)^2)$; $\Delta = 4ac - b^2$.

For the geometry specified in Fig. 2 $a = 1, b = -\frac{0,1}{w}, c = \frac{1}{w^2}$, where element width w = 50 mm

On the basis of the above-mentioned formula, the authors made a fatigue life calculation for structural elements with use of variable amplitude loading description included in Table 5. As a result of the calculation, value $N_{l_d} = 114000$ cycles is the fatigue life expressed by number of cycles.

This method has the advantage of taking into consideration physical phenomenon connected with variable amplitude loading and variable geometrical coefficient M_k. On the basis of this confirmation, this method may be of practical use for structure's element fatigue life assessment. This method can be extended in the future by solution for various Paris exponent m values.

5. Literature

[1] Kocańda S., Szala J., *Podstawy obliczeń zmęczeniowych*, PWN, Warszawa 1985.
[2] Ryżyk J.H., Gradsetyn J.S., *Tablice całek, sum, szeregów i iloczynów*, PWN, Warszawa 1964.
[3] Tomaszek H., Żurek J., Jasztal M., *Prognozowanie uszkodzeń zagrażających bezpieczeństwu lotów statków powietrznych,* Wydawnictwo naukowe ITE, Radom 2008.
[4] Kocańda D., Tomaszek H., Jasztal M., *Predicting fatigue crack growth and fatigue life under variable amplitude loading*, Fatigue of Aircraft Structures - Monographic Series Issue 2010 - Institute of Aviation Scientific Publications, Warsaw 2010.
[5] Schijve J., *The significance of fractography for investigations of fatigue crack growth under variable-amplitude loading*, Fatigue Fract Eng Mater Struct 22 (1999).

Zarys metody oceny trwałości wybranych elementów konstrukcji statku powietrznego

S t r e s z c z e n i e

W artykule zaprezentowano sposób analitycznego wyznaczenia trwałości zmęczeniowej elementów konstrukcyjnych dla przypadku obciążenia cyklami jednorodnymi oraz dla zmiennego widma obciążenia. Opis deterministyczny rozwoju pęknięcia oparto na zależności Parisa zawierającej współczynniki korekcyjne uwzględniające geometrię elementu oraz geometrię pęknięcia, a także zjawiska związane z oddziaływaniem zmiennego widma obciążenia. Zależności końcowe na trwałość zmęczeniową zostały wyznaczone dla dwóch przypadków szczególnych, gdy wykładnik równania Parisa m = 2 oraz m = 4. Przedstawiony przykład obliczeniowy pozwolił na przeprowadzenie weryfikacji liczbowej opracowanego modelu oraz zobrazował aplikacyjny charakter opracowanej metody.