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Mixture of distributions as a lifetime distribution of a technical object

K e y w o r d s

Failures, failure rate function, mixture of distributions.

S ł o w a k l u c z o w e

Uszkodzenia, funkcja intensywności uszkodzeń, mieszanina rozkładów.

S u m m a r y

The lifetime distribution is very important in reliability studies. The shape of lifetime distribution can vary considerably; therefore, it frequently cannot to approximated by simple distribution functions. This article is connected with the problem of finding of lifetime distribution with a unimodal failure rate function. For this purpose, the mixture of two distributions has been considered.

We show that a unimodal failure rate function can be obtained as a failure rate function of the mixture of an exponential and Rayleigh distributions. The numerical examples are also provided to illustrate the practical impact of this approach.

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1. Introduction

An important topic in the field of lifetime data analysis is to select the most appropriate lifetime distribution. This distribution describes the time to failure of a component, subsystem, or system. Some number of the failures are results from natural wear of the machines, while other failures may be caused by inefficient repair of the previous failures. They result from incorrect organisation of the repairs. The analysis of the results of the operation and maintenance investigations regarding the moments the failures occur prove that the set of the failures may be divided into two subsets of the primary and secondary failures. The population of times to failure is heterogeneous. The resulting population of lifetimes can be described using the statistical concept of a mixture.

The analysis of the empirical data (the length of the time intervals between the failures) indicates that it is reasonable to describe the probability distribution of the correct work times with density function f(t) as follows:

$$f(t) = p\,\lambda\, e^{-\lambda t} + (1 - p)\, f_2(t) \tag{1}$$

where $\lambda > 0$, $0 < p < 1$ and $f_2(t)$ is unknown density. This model was proposed in paper [17]. The density f(t) is a mixture of an exponential distribution and a distribution with density function $f_2(t)$.

In this paper, we study the mixture of an exponential distribution and a Rayleigh distribution. A purpose of this paper is construction of a mixture of distribution with an unimodal failure rate function. The distribution with a non-monotonic failure rate function is considered in reliability theory. The distribution with a bathtub shaped failure function (BFR) belongs to such a situation. In reliability theory, the models with BFR are very useful. A brief discussion and summary for such a distribution is given in [4] and [14]. However, there are many known examples of the application of distribution with upside-down bathtub shaped (unimodal) failure rate function (UBFR). In a particular case, the unimodal failure rate function is used in [15] and [16] to analyse the lifetime of a biological population, [1] medical data, [12] data of motor bus failure, [4] and [6] optimal burn decision, [10] ageing property in reliability, and [2] social mobility. One way of generating a distribution with a non-monotone failure rate function is the mixing of standard distributions. It is commonly known that a mixture of distributions with a decreasing failure rate function (DFR) has a decreasing failure rate function (Prochan [13]). In [9], there has been given the condition under which the mixture of an exponential distribution and an IFR (increasing failure rate function) is a DFR distribution.

Mixture gamma distribution and exponential distribution studies are shown in [7]. Klutke et al. [11] has studied the mixture of Weibull distributions and suggest that the this mixture can be a distribution with a unimodal failure rate function. However, in [19], the failure rate function has a decreasing initial period. The mixture of the two Weibull distributions has been studied in [18]. The same values of the scale parameter have given all possible types of shape failure rate functions and, for the different scale parameters, numerical computing is performed. Block et al. [5] has studied the mixture of two distributions with increasing linear failure rate functions.

Section 2 concerns a model of the mixture of distributions. In Section 3, we consider numerical examples with technical data.

2. The model of mixture distributions

We consider a mixture of the lifetimes T_1 and T_2 with the densities $f_1(t)$, $f_2(t)$, the reliability functions $R_1(t)$, $R_2(t)$, the failure rate functions $\lambda_1(t)$, $\lambda_2(t)$ and weights p and 1 – p, where $0 < p < 1$. The mixed density function is then written as

$$f(t) = p\, f_1(t) + (1 - p)\, f_2(t)$$

and the reliability function is

$$R(t) = p\, R_1(t) + (1 - p)\, R_2(t)$$

The failure rate function of mixture can be written as the mixture [3]

$$\lambda(t) = w(t)\, \lambda_1(t) + (1 - w(t))\, \lambda_2(t)$$

where $w(t) = p\, R_1(t)/R(t)$. Moreover, from [3], we have, under some mild conditions, that

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \min\{\lambda_1(t), \lambda_2(t)\}$$

In the following proposition, we give properties for the failure rate function of mixture.

Proposition 1: For the first derivative of w(t), we have

$$w'(t) = w(t)\, (1 - w(t))\, (\lambda_2(t) - \lambda_1(t))$$

Proposition 2: For the first derivative of λ(t), we obtain

$$\lambda'(t) = (1 - w(t))\, (-w(t)\, (\lambda_2(t) - \lambda_1(t))^2 + \lambda'_2(t)) + w(t)\, \lambda'_1(t)$$

Proposition 3: If $R_1(t) = \exp(-\lambda_1 t)$, for $t \geq 0$, then

$$\lambda'(t) = (1 - w(t))\,(-w(t)\,(\lambda_2(t) - \lambda_1)^2 + \lambda'_2(t))$$

Let $\lambda_1(t) = \lambda$, $\lambda_2(t) = at + b$, where $a > 0$, $b \geq 0$. The cumulative failure rate function is

$$\Lambda_2(t) = \frac{1}{2} a\, t^2 + b\, t$$

and the reliability function

$$R_2(t) = \exp\{-\frac{1}{2} a\, t^2 - b\, t\}$$

Let $h_1(t) = \lambda'_2(t) = a$, $h_2(t) = w(t)\,(\lambda_2(t) - \lambda)^2$. We will consider two cases: $\lambda \leq b$ and $\lambda > b$.

Case A: $\lambda \leq b$.

In this case the function $h_2(t)$ is increasing from $h_2(0) = p(b - \lambda)^2$ to ∞. If $a \leq p(b - \lambda)^2$ then $h_2(t) > h_1(t)$, and $\lambda'(t) < 0$. In this case $T \in$ DFR.

If $a > p(b - \lambda)^2$ then the equation $h_2(t) = h_1(t)$ has one solution. In this case, the failure rate $\lambda(t)$ is unimodal.

Case B: $\lambda > b$.

In this case, there is $t_1 = (\lambda - b)/a$ such that $h_2(t_1) = 0$. The function $h_2(t)$ is decreasing on $(0, t_1)$, and increasing on (t_1, ∞). If $p(b - \lambda)^2 \geq a$, then the equation $h_2(t) = h_1(t)$ has exactly one solution t_2 such that $t_2 > t_1$. Hence, $\lambda(t)$ is unimodal.

If $p(b - \lambda)^2 < a$, then the equation $h_2(t) = h_1(t)$ has exactly two solutions t_3 and t_4 such that $0 < t_3 < t_1$ and $t_1 < t_4$. In this case the failure rate function $\lambda(t)$ of the mixture is decreasing on $(0, t_3)$, increasing on (t_3, t_4), and decreasing on (t_4, ∞). This failure rate function we describe as a modified unimodal. We showed the following:

Proposition 4: If $b - \sqrt{a/p} < \lambda \leq b$ or $\lambda \geq b + \sqrt{a/p}$ then the failure rate function $\lambda(t)$ of the mixture (1) is unimodal.

3. Numerical examples

In this section, numerical examples are given to illustrate this model.

Example 1. We assume that $a = 0.5$, $b = 1$, $\lambda = 2$, $p \in \{0.125, 0.25, 0.375, 0.5, 0.625\}$. Fig. 1 shows a graphics of the failure rate function for this example. For $p = 0.625$, we have the modified unimodal failure rate function and for remaining values of p unimodal shape.

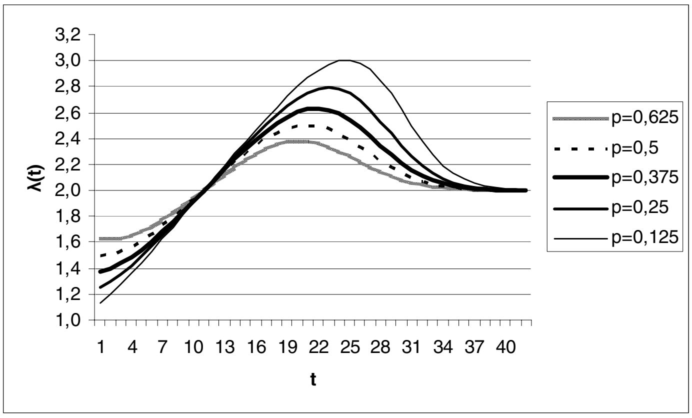

Fig. 1. The failure rate function for Example 1
Rys. 1. Funkcja intensywności dla przykładu 1

Example 2. In this example, we consider a real lifetime data. The object of the investigation is a real municipal bus transport system within a large agglomeration. The analysed system operates and maintains 210 municipal buses of various manufacturers and types. For investigation purposes, 35 buses of the same make were selected. The data set contains n = 2700 times between successive failures of the electrical system of the bus.

By maximising the logarithm of the likelihood function for grouped data, we have estimated the values of the parameters a, b, λ and p of the reliability function

$$R(t; a, b, \lambda, p) = p\, \lambda \exp(-\lambda\, t) + (1 - p) \exp(-0.5\, a\, t^2 - b\, t) \qquad (2)$$

Values of theses parameters are the following: a = 3.6476, b = 0.4495, $\lambda = 0.06813$, p = 0.6756.

We prove Kolmogorow's test of fit and compute the associated p–value, p–value = 0.14. The reliability function (2) sufficiently and precisely describes the empirical reliability function. Fig. 2 shows the failure rate function for Example 2.

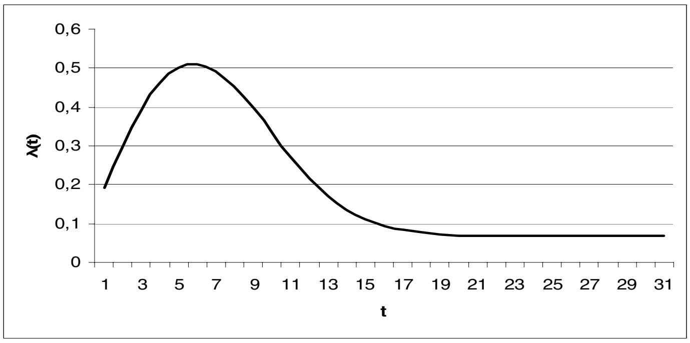

Fig. 2. The failure rate function for Example 2
Rys. 2. Funkcja intensywności dla przykładu 2

4. Conclusions

The basic idea discussed in this article is the application of the mixture of two standard distributions. In this paper, we study and attempt to determine the shape as well as the overall behaviour of the failure rate function of a mixture from two subpopulations, the exponential and Rayleigh distributions. This mixture can be used for the construction of the lifetime distribution of a technical object. The numerical example for the lifetime of an electrical system of a bus shows that a mixture can be useful for practical applications.

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Mieszanina rozkładów jako rozkład czasów życia obiektu technicznego

S t r e s z c z e n i e

Rozkłady czasów życia są bardzo ważne w badaniach niezawodnościowych. Kształt dystrybuanty czasu życia można badać dokładnie i wtedy często nie można go aproksymować przez proste rozkłady.

Pokazujemy, że jednomodalną funkcję intensywności uszkodzeń można otrzymać jako funkcję intensywności uszkodzeń mieszaniny rozkładu wykładniczego i rozkładu Rayleigha. W celu pokazania praktycznego znaczenia tego podejścia podano przykłady numeryczne.