Operation of an aircraft with risk of its loss

Key words
Reliability, risk, intensity, probability.

Summary
The subject of this paper concerns the way of an aircraft operation in which there are three states: the operational state “1”, the state of repair “2”, and the state of a complete loss of airworthiness “3”. Transition between states results from the intensity of damages and repairs and the intensity of reaching the state of a complete lack of airworthiness. The paper assumes three diagrams of possible transitions between states. The assumed diagrams were supplemented with sets of difference equations, which were transformed into sets of differential equations. Their solutions provided dependencies for the probability of operational state $P_1(t)$, the probability of repair $P_2(t)$, and the probability of a complete loss of airworthiness $P_3(t)$. The obtained dependencies enabled preparation of graphs, which is helpful in interpreting the safety of flights of an aircraft.

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1. Introduction

Deliberations on the safety of flying are the reason for the need of analysing different methods of the operation of aircraft. This paper analyses three ways of operating aircraft. These ways include the following states:

- The state of reliability “1”: the probability \( P_1(t) \) describes an aircraft in this state;
- The state of repair “2”: the probability \( P_2(t) \) describes an aircraft in this state; and,
- The state of a complete lack of airworthiness “3”: the probability \( P_3(t) \) describes an aircraft in this state.

There are three variants of operation:

- The first variant presented in Figure 1.
- The second variant presented in Figure 2.
- The third variant presented in Figure 3.

Possible transitions between the states are symbolised by arrows. A driving force of a transition between states is the intensity of damages and repairs, and the intensity of transition to the state of a complete lack of airworthiness.

The following issues were presented for the above-mentioned ways of operating aircraft:

- Sets of difference equations;
- Sets of differential equations; and,
- Solutions of sets of equations for individual variants.

The obtained dependencies of probabilities for individual variants were used to present their graph in the function of time.

2. The introduction of equation sets for the assumed diagrams of operation

The first variant

In this diagram of an aircraft operation, the considered issue concerns the transition from the operational state “1” to the state of repair “2” with the intensity \( \lambda_1 \); on the other hand, there is possibility of transition from the state of repair “2” to the operational state “1” with the intensity \( \mu \) or to the state of a complete lack of airworthiness “3” with the intensity \( \lambda_2 \). The state of a complete lack of airworthiness is an absorbing state. Figure 1 presents a graph of transitions between the states. The graph includes the above-mentioned quantities and probabilities of staying in the following states: \( P_1(t) \), \( P_2(t) \), \( P_3(t) \).

![Fig. 1. Operation diagram according to the first variant](image-url)
The following set of difference equations can be arranged for such an operation diagram:

\[
\begin{align*}
P_1(t + \Delta t) &= (1 - \lambda_1 \Delta t)P_1(t) + \mu \Delta t P_2(t) \\
P_2(t + \Delta t) &= \lambda_1 \Delta t P_1(t) + (1 - \mu \Delta t)(1 - \lambda_2 \Delta t)P_2(t) \\
P_3(t + \Delta t) &= P_3(t) + \lambda_2 \Delta t P_2(t).
\end{align*}
\]

(1)

For the first variant of operation, after transformations, we obtain the following set of equations:

\[
\begin{align*}
P'_1(t) &= -\lambda_1 P_1(t) + \mu P_2(t) \\
P'_2(t) &= \lambda_1 P_1(t) - (\mu + \lambda_2) P_2(t) \\
P'_3(t) &= \lambda_2 P_2(t).
\end{align*}
\]

(2)

The second variant

This diagram of operation includes the possibility of transition from the operational state “1” to the state of a complete lack of airworthiness with the intensity \( \lambda_2 \); however, it does not include the possibility of transition from the state of repair “2” to the state of a complete lack of airworthiness “3”. Figure 2 presents a graph of transitions between the states, other notations are the same as the above.

Fig. 2. Operation diagram according to the second variant
Rys. 2. Schemat eksploatacji według drugiego wariantu

In this variant, we can also arrange a set of difference equations illustrating transition between the states.

\[
\begin{align*}
P_1(t + \Delta t) &= (1 - \lambda_1 \Delta t)(1 - \lambda_2 \Delta t)P_1(t) + \mu \Delta t P_2(t) \\
P_2(t + \Delta t) &= \lambda_1 \Delta t P_1(t) + (1 - \mu \Delta t)P_2(t) \\
P_3(t + \Delta t) &= P_3(t) + \lambda_2 \Delta t P_2(t).
\end{align*}
\]

(3)
In the second variant, after transformations of the set (3), we obtain the following set of differential equations.

\[
\begin{align*}
P_1'(t) &= -(\lambda_1 + \lambda_2)P_1(t) + \mu P_2(t) \\
P_2'(t) &= \lambda_1 P_1(t) - \mu P_2(t) \\
P_3'(t) &= \lambda_2 P_2(t)
\end{align*}
\]  

(4)

**The third variant**

We shall consider the third way of operation.

This diagram of operation includes the possibility of transition from the operational state “1” to the state of a complete lack of airworthiness “3” with the intensity \(\lambda_2\) and the possibility of transition from the state of repair “2” to state of a complete lack of airworthiness “3” with the intensity \(\lambda_3\). Figure 3 presents a graph of transitions between the states, other notations are the same as the above. This variant includes both the above-described variants.

\[
\begin{align*}
P_1(t + \Delta t) &= (1 - \lambda_1 \Delta t)(1 - \lambda_2 \Delta t)P_1(t) + \mu \Delta t P_2(t) \\
P_2(t + \Delta t) &= \lambda_1 \Delta t P_1(t) + (1 - \lambda_2 \Delta t)(1 - \mu \Delta t)P_2(t) \\
P_3(t + \Delta t) &= P_3(t) + \lambda_2 \Delta t P_1(t) + \lambda_3 \Delta t P_2(t).
\end{align*}
\]  

(5)

From the set (5), we obtained the following set of differential equations

\[
\begin{align*}
P_1'(t) &= -(\lambda_1 + \lambda_2)P_1(t) + \mu P_2(t) \\
P_2'(t) &= \lambda_1 P_1(t) - (\mu + \lambda_3)P_2(t) \\
P_3'(t) &= \lambda_2 P_1(t) + \lambda_3 P_2(t)
\end{align*}
\]  

(6)
For all variants, initial conditions are the same:

for \( t = 0 \rightarrow P_1(0) = 1, \ P_2(0) = 0, \ P_3(0) = 0. \)

3. Solution of the set of equations for the accepted conditions

The first variant

In this variant, we obtained the following set of differential equations:

\[
\begin{align*}
P_1'(t) &= -\lambda_1 P_1(t) + \mu P_2(t) \\
P_2'(t) &= \lambda_3 P_1(t) - (\mu + \lambda_2)P_2(t) \\
P_3'(t) &= \lambda_2 P_2(t)
\end{align*}
\]  

(7)

As a result of calculations, we obtained [4]:

\[
\begin{align*}
P_1(t) &= \frac{r_1 + \lambda_1}{r_1 - r_2} e^{rt} - \frac{r_2 + \lambda_2}{r_1 - r_2} e^{rt} \\
P_2(t) &= \frac{(r_1 + \lambda_1)(r_2 + \lambda_4)}{\mu(r_1 - r_2)} \left[ e^{rt} - e^{-rt} \right] \\
P_3(t) &= \frac{\lambda_2(r_1 + \lambda_1)(r_2 + \lambda_4)}{\mu(r_1 - r_2)} \left[ \frac{1}{r_2 e^{rt} - \frac{1}{r_1 e^{rt}}} \right] - \frac{\lambda_2}{\mu r_2 r_2} (r_2 + \lambda_4) (r_1 + \lambda_1)
\end{align*}
\]  

(8)

where:

\[
\begin{align*}
r_1 &= \frac{-\left(\lambda_1 + \lambda_2 + \mu\right) - \sqrt{\left(\lambda_1 - \lambda_2\right)^2 + 2\mu(\lambda_1 + \lambda_2) + \mu^2}}{2}, \\
r_2 &= \frac{-\left(\lambda_1 + \lambda_2 + \mu\right) + \sqrt{\left(\lambda_1 - \lambda_2\right)^2 + 2\mu(\lambda_1 + \lambda_2) + \mu^2}}{2}.
\end{align*}
\]  

(9)

The above-presented solution meets the initial conditions, and the sum of determined probabilities equals the 1. The only thing to do is check a graph of the determined functions for large \( t (t \rightarrow \infty) \). This is included in the paper [4].

The second variant

For this variant of an aircraft operation, the following set of differential equations was introduced:
As a result of calculations, we obtained:

\[
\begin{align*}
P'_1(t) &= -(\lambda_1 + \lambda_2)P_1(t) + \mu P_2(t) \\
P'_2(t) &= \lambda_1 P_1(t) - \mu P_2(t) \\
P'_3(t) &= \lambda_2 P_1(t)
\end{align*}
\]  
(10)

where:

\[
\begin{align*}
P_1(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} \frac{e^{rt}}{r_1} + \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{rt} \\
P_2(t) &= -\frac{(r_1 + \lambda_1 + \lambda_2)(r_2 + \lambda_1 + \lambda_2)}{\mu(r_1 - r_2)} \left[ e^{rt} - e^{r't} \right] \\
P_3(t) &= \frac{\lambda_2}{r_1 - r_2} \left[ -\frac{r_2 + \lambda_1 + \lambda_2}{r_1} \frac{e^{rt}}{r_1} + \frac{r_1 + \lambda_1 + \lambda_2}{r_2} e^{r't} \right] + 1,
\end{align*}
\]  
(11)

where:

\[
\begin{align*}
&\frac{r_1}{2} = -\frac{(\lambda_1 + \lambda_2 + \mu)}{\sqrt{\lambda_1 + \lambda_2}}^2 + 2\mu(\lambda_1 - \lambda_2) + \mu^2, \\
&\frac{r_2}{2} = -\frac{(\lambda_1 + \lambda_2 + \mu)}{\sqrt{\lambda_1 + \lambda_2}}^2 + 2\mu(\lambda_1 - \lambda_2) + \mu^2.
\end{align*}
\]  
(12)

The above-presented solution meets the initial conditions, and the sum of determined probabilities equals the 1. The only thing to do is check a graph of the determined functions for large \( t (t \to \infty) \).

\[
\begin{align*}
\lim_{t \to \infty} P_1(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} \lim_{t \to \infty} e^{rt} + \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} \lim_{t \to \infty} e^{r't} \\
\lim_{t \to \infty} P_2(t) &= -\frac{(r_1 + \lambda_1 + \lambda_2)(r_2 + \lambda_1 + \lambda_2)}{\mu(r_1 - r_2)} \left[ \lim_{t \to \infty} e^{rt} - \lim_{t \to \infty} e^{r't} \right] \\
\lim_{t \to \infty} P_3(t) &= \frac{\lambda_2}{r_1 - r_2} \left[ -\frac{r_2 + \lambda_1 + \lambda_2}{r_1} \lim_{t \to \infty} e^{rt} + \frac{r_1 + \lambda_1 + \lambda_2}{r_2} \lim_{t \to \infty} e^{r't} \right] + 1.
\end{align*}
\]

Because:

\[
\begin{align*}
&\frac{r_1}{2} = -\frac{(\lambda_1 + \lambda_2 + \mu)}{\sqrt{\lambda_1 + \lambda_2}}^2 - 4\mu\lambda_2 < 0, \\
&\frac{r_2}{2} = -\frac{(\lambda_1 + \lambda_2 + \mu)}{\sqrt{\lambda_1 + \lambda_2}}^2 - 4\mu\lambda_2 < 0,
\end{align*}
\]
so
\[
\lim_{t \to \infty} e^{\eta t} = 0, \\
\lim_{t \to \infty} e^{\xi t} = 0.
\]

And consequently,
\[
\lim_{t \to \infty} P_1(t) = 0, \\
\lim_{t \to \infty} P_2(t) = 0, \\
\lim_{t \to \infty} P_3(t) = 1.
\]

Ultimately, we obtained:
\[
\begin{align*}
P_1(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{\eta t} + \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{\xi t} \\
P_2(t) &= -\frac{(r_1 + \lambda_1 + \lambda_2)(r_2 + \lambda_1 + \lambda_2)}{\mu(r_1 - r_2)} [e^{\eta t} - e^{\xi t}] \\
P_3(t) &= \frac{\lambda_2}{r_1 - r_2} \left[ -\frac{r_2 + \lambda_1 + \lambda_2}{r_1} e^{\eta t} + \frac{r_1 + \lambda_1 + \lambda_2}{r_2} e^{\xi t} \right] + 1,
\end{align*}
\]

where:
\[
\begin{align*}
r_1 &= -\left( \lambda_1 + \lambda_2 + \mu \right) - \sqrt{(\lambda_1 + \lambda_2)^2 + 2\mu(\lambda_1 - \lambda_2) + \mu^2}, \\
r_2 &= -\left( \lambda_1 + \lambda_2 + \mu \right) + \sqrt{(\lambda_1 + \lambda_2)^2 + 2\mu(\lambda_1 - \lambda_2) + \mu^2}.
\end{align*}
\]

The probability \(P_1(t)\) is a decreasing function of time and adopts values from 1 to 0. On the other hand, the probability \(P_3(t)\) is an increasing function and adopts values from 0 to 1. The probability \(P_2(t)\) has one specific extremum (maximum) in the following point:
\[
t = t_{\text{extr}} = \frac{1}{r_2 - r_1} \ln \frac{r_1}{r_2},
\]
of the following value:

\[
P_2(t_{\text{err}}) = \frac{\lambda_1}{\sqrt{\mu \lambda_2}} \left( \sqrt{(\lambda_1 + \lambda_2 + \mu)^2 + 2\mu(\lambda_1 - \lambda_2) + \mu^2} \right) e^{-\left(\lambda_1 + \lambda_2 + \mu\right) t}.
\]

(16)

The third variant

In this variant, transition between the states is described by the following set of differential equations:

\[
\begin{align*}
P_1'(t) &= -\left(\lambda_1 + \lambda_2\right)P_1(t) + \mu P_2(t) \\
P_2'(t) &= \lambda_1 P_1(t) - \left(\mu + \lambda_3\right)P_2(t) \\
P_3'(t) &= \lambda_2 P_1(t) + \lambda_3 P_2(t).
\end{align*}
\]

(17)

As a result of calculations, we obtained:

\[
\begin{align*}
P_1(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{\mu t} + \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{\mu t} \\
P_2(t) &= \frac{r_2 + \lambda_1 + \lambda_2}{(r_1 - r_2) \mu} \left( e^{\mu t} + e^{\mu t} \right) \\
P_3(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} \left( \frac{r_1 + \lambda_1 + \lambda_2}{\mu r_1} + \frac{\lambda_2 \mu}{\mu r_2} + 1, \right)
\end{align*}
\]

(18)

where:

\[
\begin{align*}
r_1 &= -\frac{\left(\lambda_1 + \lambda_2 + \lambda_3 + \mu\right) - \sqrt{(\lambda_1 + \lambda_2 - \lambda_3 - \mu)^2 + 2\mu \lambda_1}}{2}, \\
r_2 &= -\frac{\left(\lambda_1 + \lambda_2 + \lambda_3 + \mu\right) + \sqrt{(\lambda_1 + \lambda_2 - \lambda_3 - \mu)^2 + 2\mu \lambda_1}}{2}.
\end{align*}
\]

(19)

The above-presented solution meets the initial conditions, and the sum of determined probabilities equals 1. The only thing to do is check a graph of the determined functions for large \( t (t \to \infty) \).
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\[
\lim_{t \to \infty} P_1(t) = -\frac{\lambda_1 + \lambda_2}{r_1 - r_2} \lim_{t \to \infty} e^{r_1 t} + \frac{\lambda_3}{r_1 - r_2} \lim_{t \to \infty} e^{r_2 t},
\]

\[
\lim_{t \to \infty} P_2(t) = \left( \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} \right) \frac{r_1 + \lambda_1 + \lambda_2}{\mu \lambda_1 + \lambda_2} \left( \lim_{t \to \infty} e^{r_1 t} + \lim_{t \to \infty} e^{r_2 t} \right),
\]

\[
\lim_{t \to \infty} P_3(t) = -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} \left( \frac{r_1 \lambda_1 + \lambda_2}{\mu} \right) \left( \frac{\lambda_1 + \lambda_2}{r_1 - r_2} \right) \left( \frac{\lambda_1 + \lambda_2}{\mu \lambda_1 + \lambda_2} \right) \lim_{t \to \infty} e^{r_1 t} + 1.
\]

Because:

\[
r_1 = -\frac{\lambda_1 + \lambda_2 + \lambda_3 + \mu}{2} - \sqrt{\left(\frac{\lambda_1 + \lambda_2 - \lambda_3 - \mu}{4} + 4 \frac{\mu \lambda_1}{2}\right)} < 0,
\]

\[
r_2 = -\frac{\lambda_1 + \lambda_2 + \lambda_3 + \mu}{2} + \sqrt{\left(\frac{\lambda_1 + \lambda_2 - \lambda_3 - \mu}{4} + 4 \frac{\mu \lambda_1}{2}\right)} < 0,
\]

so

\[
\lim_{t \to \infty} e^{r_1 t} = 0,
\]

\[
\lim_{t \to \infty} e^{r_2 t} = 0.
\]

And consequently:

\[
\lim_{t \to \infty} P_1(t) = 0,
\]

\[
\lim_{t \to \infty} P_2(t) = 0,
\]

\[
\lim_{t \to \infty} P_3(t) = 1.
\]

Ultimately, we obtained:

\[
\begin{align*}
\left\{ 
P_1(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} e^{r_1 t} + \frac{\lambda_3}{r_1 - r_2} e^{r_2 t} \\
P_2(t) &= \left( \frac{r_1 + \lambda_1 + \lambda_2}{r_1 - r_2} \right) \frac{r_1 + \lambda_1 + \lambda_2}{\mu \lambda_1 + \lambda_2} \left( e^{r_1 t} + e^{r_2 t} \right) \\
P_3(t) &= -\frac{r_2 + \lambda_1 + \lambda_2}{r_1 - r_2} \left( \frac{r_1 \lambda_1 + \lambda_2}{\mu} \right) \left( \frac{\lambda_1 + \lambda_2}{r_1 - r_2} \right) \left( \frac{\lambda_1 + \lambda_2}{\mu \lambda_1 + \lambda_2} \right) e^{r_1 t} + 1,
\end{align*}
\]
where:
\[
r_1 = -\frac{1}{2} \left( \lambda_1 + \lambda_2 + \lambda_3 + \mu - \sqrt{(\lambda_1 + \lambda_2 - \lambda_3 - \mu)^2 + 4\mu\lambda_1} \right),
\]
\[
r_2 = -\frac{1}{2} \left( \lambda_1 + \lambda_2 + \lambda_3 + \mu + \sqrt{(\lambda_1 + \lambda_2 - \lambda_3 - \mu)^2 + 4\mu\lambda_1} \right).
\]

The probability \( P_1(t) \) is a decreasing function of time and adopts values from 1 to 0. On the other hand, the probability \( P_3(t) \) is an increasing function and adopts values from 0 to 1. The probability \( P_2(t) \) has one specific extremum (maximum) in the following point:

\[
t = t_{\text{extr}} = \frac{1}{r_2 - r_1} \ln \frac{r_1}{r_2},
\]

of the following value:

\[
P_2(t_{\text{extr}}) = \frac{\lambda_1}{\sqrt{(\lambda_1 + \lambda_2)(\lambda_3 + \mu) - \mu\lambda_1}} \left( \frac{\lambda_1 + \lambda_2 + \lambda_3 + \mu}{\sqrt{(\lambda_1 + \lambda_2 - \lambda_3 - \mu)^2 + 4\mu\lambda_1}} + \frac{\lambda_1 + \lambda_2 - \lambda_3 - \mu + 4\mu\lambda_1}{2\sqrt{(\lambda_1 + \lambda_2)(\lambda_3 + \mu) - \mu\lambda_1}} \right).
\]

4. Graphic presentation of results

The further part includes graphs of the changes of probabilities calculated in particular variants. The graphs aim at showing the character of these changes. The graphs were developed for hypothetical data:

Variant 1: \( \lambda_1 = 0.009 : \lambda_2 = 0.008 : \mu = 0.01 \).
Variant 2: \( \lambda_1 = 0.009 : \lambda_2 = 0.008 : \mu = 0.01 \).
Variant 3: \( \lambda_1 = 0.009 : \lambda_2 = 0.007 : \lambda_3 = 0.001 : \mu = 0.01 \).

A unit of intensity is the inverse of the accepted unit of time.
Fig. 4. Change of probabilities according to the first variant
Rys. 4. Zmiana prawdopodobieństwa wg wariantu pierwszego

Fig. 5. Change of probabilities according to the second variant
Rys. 5. Zmiana prawdopodobieństwa wg wariantu drugiego

Fig. 6. Change of probabilities according to the third variant
Rys. 6. Zmiana prawdopodobieństwa wg wariantu trzeciego
Fig. 7. Change of the probability $P_1(t)$ according to the variants
Rys. 7. Zmiana prawdopodobieństwa $P_1(t)$ wg wariantów

Fig. 8. Change of the probability $P_2(t)$ according to the variants
Rys. 8. Zmiana prawdopodobieństwa $P_2(t)$ wg wariantów

Fig. 9. Change of the probability $P_3(t)$ according to the variants
Rys. 9. Zmiana prawdopodobieństwa $P_3(t)$ wg wariantów
5. Final remarks

Data used for graphic presentation and results of calculations are selected randomly. Characters of the graphs of probabilities for all variants are similar. Figure 7 presents a graph of the probability \( P_1(t) \) for the assumed variants. The graph shows that differences in values of this probability are small for the second and third variant. The probability \( P_1(t) \) for the first variant differs markedly from the other variants.

It shall be stressed that the analysis of the operation of aircraft according to the assumed diagrams is simplified, if particular intensities are considered as constant values.

The above-presented results can be used to examine the safety of flights of aircraft with the use of the renewal theory. In this case, the probability \( P_1(t) \) will function as readiness coefficient. The example of such deliberations is presented in the paper [4].

Literature


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Eksploatacja statku powietrznego z ryzykiem jego utraty

Przedmiotem pracy jest sposób eksploatacji statku powietrznego, w którym wyróżnia się stan zdolności „1”, stan naprawy „2” i stan całkowitej utraty zdolności „3”. Przejście między stanami następuje w wyniku intensywności uszkodzeń, naprawy i intensywności osiągania stanu całkowitej niezdolności.

W pracy przyjęto trzy schematy możliwych przejść między stanami. Dla tak przyjętych schematów ułożono układy równań różniczkowych, z których po przekształceniu otrzymano układy równań różniczkowych.

Rozwiązując układy równań otrzymano zależności na prawdopodobieństwo zdolności \( P_1(t) \), prawdopodobieństwo przebywania statku w naprawie \( P_2(t) \) i prawdopodobieństwo całkowitej utraty zdolności \( P_3(t) \).

Otrzymane zależności pozwoliły na sporządzenie wykresów, co ułatwia interpretację bezpieczeństwa lotów statku powietrznego.